Lecture 2

3. The Tangent Space

Definition 3.1 A tangent vector at $p \in M$ is a function

$$v_p : C^\infty(M) \longrightarrow \mathbb{R}$$

such that

$$v(af + g) = a\,v(f) + b\,v(g)$$

for all $a \in \mathbb{R}$ and $f, g \in C^\infty(M)$, and

$$v_p(fg) = g(p)\,v(f) + f(p)\,v(g)$$

for all $f, g \in C^\infty(M)$.

The tangent space at p is the set of all tangent vectors at p and is denoted by T_pM.

Example 3.2 Let (φ, U) be a chart that contains $p \in M$. Then*

$$\partial_i|_p(f) := \partial_i(f \circ \varphi^{-1})(\varphi(p)) = \frac{\partial}{\partial x^i}(f \circ \varphi^{-1}(x))\Big|_{x=\varphi p}$$

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* On the RHS, $\partial_i = \frac{\partial}{\partial x^i}$ is the usual partial derivative defined by the Cartesian coordinates $x = (x^1, \ldots, x^n)$ on $\mathbb{R}^n$ that cover the image of the chart $\varphi(U) \subset \mathbb{R}^n$.

is a tangent vector at p. (Hw: Verify this statement).

Lemma 3.3 Suppose $V_p \in T_pM$.

(i) If $f, g \in C^\infty(M)$ are equal on an open neighborhood of p, then

$$V_p(f) = V_p(g).$$

(ii) If $h \in C^\infty(M)$ is constant on an open neighborhood of p, then

$$V_p(h) = 0.$$

Proof

(i) Suppose U is an open neighborhood of p and that

$$f|U = g|U$$

Then

$$h := f - g$$

satisfies

$$h|U = 0$$

Next let $\chi \in C^\infty(M)$ be a cutoff function satisfying

$$0 \le \chi \le 1, \quad \operatorname{supp}\chi \subset U \quad \text{and} \quad \chi|V = 1$$

where $V \subset U$ is an open neighborhood of p. Then $\chi h = 0$ and we that

$$V_p(\chi h) = 0$$

follows by linearity of the map $V_p : C^\infty(M) \to \mathbb{R}$.

By the Leibnitz property of tangent vectors, we get that

$$0 = V_p(\chi h) = \chi(p) V_p(h) + h(p) V_p(\chi),$$

($\chi(p) = 1$, $h(p) = 0$)

and so

$$V_p(h) = 0 \iff V_p(s) = V_p(g).$$

(ii) Suppose that $h \in C^\infty(M)$ is constant in an open neighborhood of $p \in M$. Then by (i) we can, without loss of generality, assume that

$$h = c \quad \text{on } M$$

for some $c \in \mathbb{R}$. Noting that

$$V_p(1) = V_p(1 \cdot 1) = 1 \, V_p(1) + 1 \, V_p(1) = 2 V_p(1),$$

we see that

$$V_p(1) = 0.$$

Since

$$h = c \cdot 1,$$

it follows that

$$V_p(h) = V_p(c \cdot 1) = c\, V_p(1) = 0. \qquad \blacksquare$$

Theorem 3.4 Suppose that (U, φ) is a chart containing $p \in M$. Then

$$\{ \partial_1|_p, \ldots, \partial_n|_p \}$$

is a basis of T_pM. Moreover,

$$V_p = V_p(\varphi^i)\, \partial_i|_p \qquad (\varphi = (\varphi^1, \ldots, \varphi^n))$$

for all $V_p \in T_pM$.

Proof

Without loss of generality, we assume that

$$\varphi(p) = 0$$

and

$$\varphi(U) = B_R(0)$$

for some $R > 0$. Next, suppose that $f \in C^\infty(M)$ and define

$$\tilde{f} = f \circ \varphi^{-1} \in C^\infty(B_R(0)).$$

Observing that

$$\tilde{f}(x) = f(p) + \tilde{f}_i(x)\, x^i$$

where

$$\tilde{f}_i(y) = \int_0^1 \frac{\partial f}{\partial x^i}(ty)\, dt ,$$

we can write f as

$$f = f(p) + f_i \varphi^i \qquad (3.1)$$

where

$$\varphi(q) = (\varphi^1(q), \ldots, \varphi^n(q))$$

and

$$f_i = \tilde{f}_i \circ \varphi$$

Applying $\partial_i|_p$ to (3.1) gives

$$\partial_j|_p f = \partial_j|_p f(p) + \left(\varphi^i(p)\, \partial_j|_p (f_i) + f_i(p)\, \partial_j|_p (\varphi^i) \right)$$

$$= f_i(p) \frac{\partial x^i}{\partial x^j} = f_j(p) .$$

So

$$V_p(f) = \left(V_p(f_i) \underbrace{\varphi^i(p)}_{0} + \underbrace{f_i(p)}_{\partial_i|_p f} V_p(\varphi^i) \right)$$

$$= V_p(\varphi^i)\, \partial_i|_p f ,$$

and we conclude that

$$V_p = V_p(\varphi^i)\, \partial_i|_p .$$

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Notation 3.5

Given a chart (U, φ) on a manifold M, we will also denote the tangent vector $\partial_i|_p$ by

$$\partial_i|_p = \frac{\partial}{\partial \varphi^i}\Big|_p$$

where $\varphi = (\varphi^1, \ldots, \varphi^n)$.

Given a smooth function $f \in C^\infty(M)$, we will often write

$$\frac{\partial f}{\partial \varphi_i}\Big|_p := \partial_i (f \circ \varphi^{-1})(\varphi(p))$$

4. The Tangent Map and Tangent Bundle

Definition 4.1 Suppose $\Phi : M \to N$ is a smooth map and $p \in M$. Then the linear map

$$T_p f : M \to N : v_p \longmapsto T_p(v_p)$$

defined by

$$T_p(v_p)(f) = v(f \circ \Phi) \quad \forall\, f \in C^\infty(N)$$

is called the tangent map.

Proposition 4.2 (Chain Rule)

Suppose $\Phi : M \to N$ and $\Psi : N \to P$ are smooth maps. Then

$$T_p(\Psi \circ \Phi) = T_{\Phi(p)}\Psi \circ T_p \Phi$$

Proof Suppose that $f \in C^\infty(M)$ and $v_p \in T_p M$. Then

$$T_p(\Psi\circ\Phi)(v_p)(f) = v_p(f\circ\Psi\circ\Phi)$$

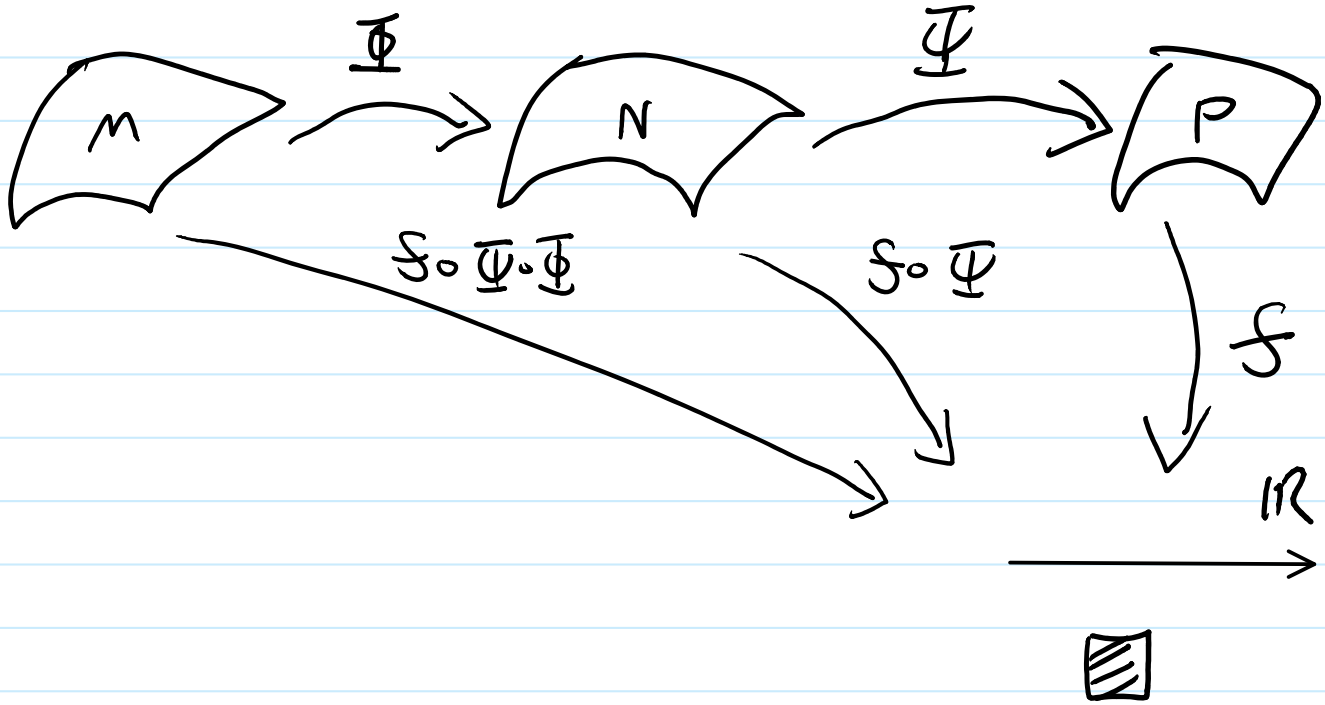
$$= T_p\Phi(v_p)(f\circ\Psi)$$

$$\in T_{\Phi(p)}M$$

$$= T_{\Phi(p)}\Psi\left(T_p\Phi(v_p)\right)$$

shows that

$$T_p(\Psi\circ\Phi) = T_{\Phi(p)}\Psi\circ T_p\Phi\ .$$

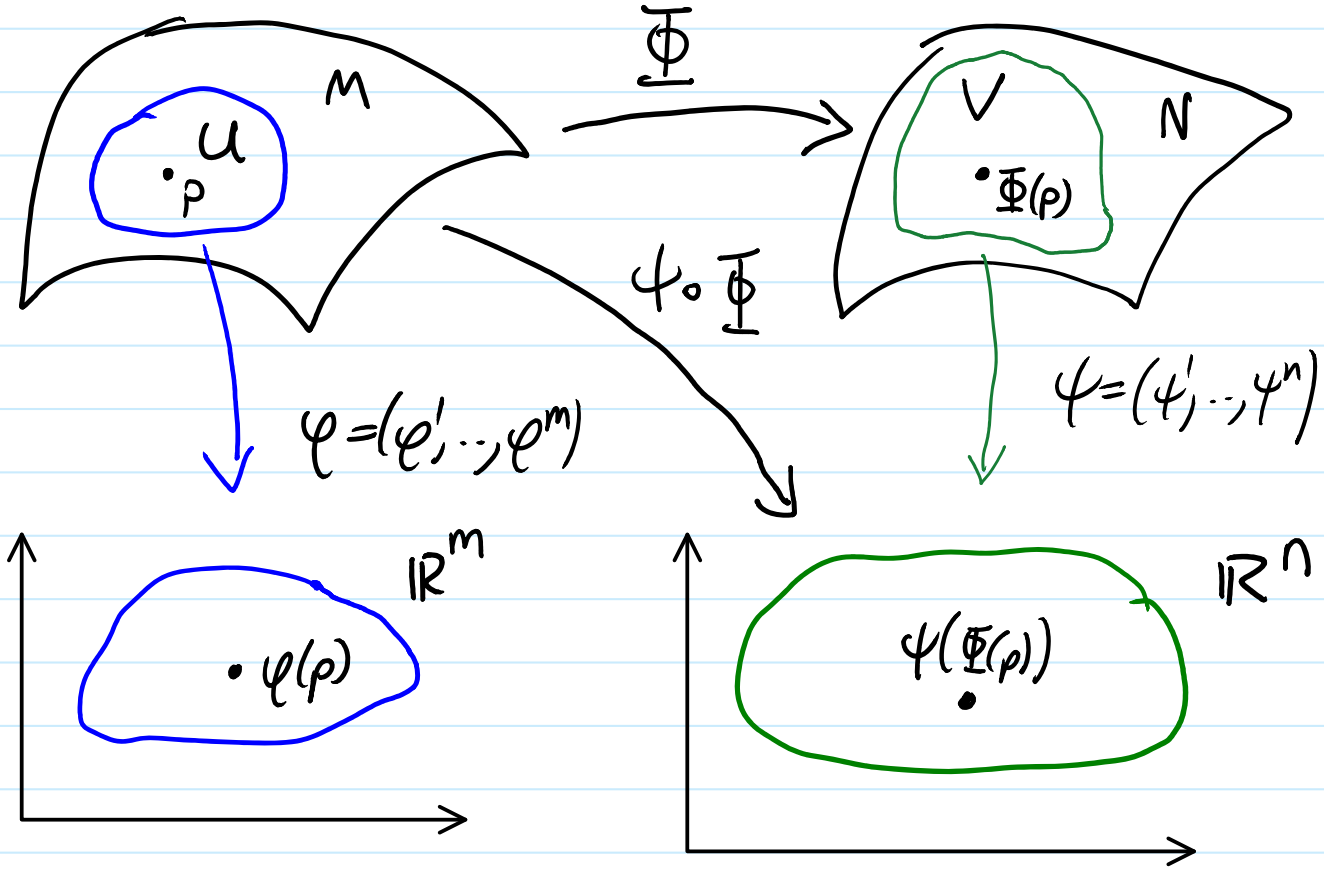


Proposition 4.3 Suppose $\Phi: M \to N$ is smooth, $(U, \varphi = (\varphi^1, \ldots, \varphi^m))$ is a chart containing $p \in M$ and $(V, \psi = (\psi^1, \ldots, \psi^n))$ is a chart containing $\Phi(p)$. Then

$$T_p\Phi\left(\frac{\partial}{\partial \varphi^j}\Big|_p\right) = \frac{\partial \Phi^i}{\partial \varphi^j}\Big|_p \frac{\partial}{\partial \psi^i}\Big|_{\Phi(p)} \qquad j = 1, \ldots, m$$

where

$$\Phi^i = \psi^i \circ \Phi .$$



Proof:

Suppose $f \in C^\infty(N)$. Then

$$T_p\Phi\left(\frac{\partial}{\partial\varphi^j}\right)(f) = \frac{\partial}{\partial\varphi^j}\Big|_p (f\circ\Phi)$$

(by def of the of the tangent map)

$$= \partial_j\left(f\circ\Phi\circ\varphi^{-1}\right)(\varphi(p)) \quad \left(\text{by def of } \frac{\partial}{\partial\varphi^j}\Big|_p\right)$$

$$= \partial_j\left(f\circ\psi^{-1}\circ\psi\circ\Phi\circ\varphi^{-1}\right)(\varphi(p))$$

$$= \partial_i\left(f\circ\psi^{-1}\right)(\psi\circ\Phi\circ\varphi^{-1}\circ\varphi(p))\ \partial_j\left(\psi^i\circ\Phi\circ\varphi^{-1}\right)(\varphi(p))$$

↑ by the ordinary chain rule from calculus

$$= \partial_i\left(f\circ\psi^{-1}\right)(\psi(\Phi(p)))\ \partial_j\left(\psi^i\circ\Phi\circ\varphi^{-1}\right)(\varphi(p))$$

$$= \frac{\partial}{\partial\psi^i}\Big|_{\Phi(p)}(f)\ \frac{\partial\Phi^i}{\partial\varphi^j}\Big|_p .$$

↑ by def of $\frac{\partial}{\partial\psi^i}\Big|_{\Phi(p)}$ and $\frac{\partial\Phi^i}{\partial\varphi^j}\Big|_p$ where $\Phi^i = \psi^i\circ\Phi$.

Since f was chosen arbitrarily, we conclude that

$$T_p\Phi\left(\frac{\partial}{\partial\varphi^j}\right) = \frac{\partial\Phi^i}{\partial\varphi^j}\Big|_p \frac{\partial}{\partial\psi^i}\Big|_{\Phi(p)} \qquad j = 1, 2, \ldots, m$$

Definition 4.4 The set

$$TM = \bigcup_{p \in M} T_pM$$

is called the tangent bundle, and the map

$$\pi : TM \longrightarrow M$$

defined by

$$T_pM \ni V_p \longmapsto \pi(V_p) = p \in M$$

is known as the projection map.

For $\Phi \in C^\infty(M, N)$, the tangent map

$$T\Phi : TM \longrightarrow TN$$

is defined by

$$T\Phi(V_p) := T_p\Phi(V_p) \quad \forall\, V_p \in T_pM,\ p \in M.$$

Proposition 4.5 (Chain rule again)

Suppose $\Phi \in C^\infty(M,N)$, $\Psi \in C^\infty(N,P)$.

Then

$$T(\Psi \circ \Phi) = T\Psi \circ T\Phi$$

Proof

Fix $v_p \in T_pM \subset TM$. Then

$$T(\Psi\circ\Phi)(v_p) = T_p(\Psi\circ\Phi)(v_p) \quad \text{by def of } T(\Psi\circ\Phi)$$

$$= T_{\Phi(p)}\Psi\left(T_p\Phi(v_p)\right) \text{ by Prop. 4.2}$$

$$= T_{\Phi(p)}\Psi\left(T\Phi(v_p)\right) \text{ by def of } T\Phi$$

$$= T\Psi\left(T\Phi(v_p)\right) \text{ by def of } T\Psi$$

$$= T\Psi \circ T\Phi\,(v_p).$$

Since this holds for all $v_p \in T_pM$ and $p \in M$, it follows that

$$T(\Psi\circ\Phi) = T\Psi \circ T\Phi$$

Exercise 4.6

If id_M is the identity map on a manifold M. Show that

$$T id_M = id_{TM},$$

where id_{TM} is the identity map on the tangent bundle TM.

Exercise 4.7

If $\Phi \in C^\infty(M,N)$ is a diffeomorphism, show that

$$T\Phi : TM \longrightarrow TN$$

is bijective and also that

$$(T\Phi)^{-1} = T(\Phi^{-1}).$$

Exercise 4.8

Suppose $U \subset \mathbb{R}^n$ is an open set. Then (U, ξ), where $\xi = id|_U$ and $id : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is the identity map, defines a global chart on U. With respect to this chart, the basis vectors

$\partial_i|_x \in T_x U$ $i = 1, 2, \ldots, n$ are well defined for each $x \in U$. Given $V_x \in T_x U$, let $V_x = V_x^i \partial_i|_x$ be the unique expansion for V_x in terms of the basis $\{\partial_i|_x\}_{i=1}^n$. Show that the map

$$\lambda : TU \longrightarrow U \times \mathbb{R}^n$$

determined by

$$\lambda(V_x) = (x, V_x^1, \ldots, V_x^n) \quad \forall V_x \in T_x U$$

is well-defined and bijective. Moreover, show that

$$\lambda|_{T_x U} : T_x U \longrightarrow \{x\} \times \mathbb{R}^n \cong \mathbb{R}^n$$

is a (linear) isomorphism for each $x \in U$.

Note An alternate approach is to define the inverse of the map λ, denoted $\check{\lambda}$, first by

$$\check{\lambda} : U \times \mathbb{R}^n \longrightarrow TU : (x, V) \longmapsto V_x$$

where $V_x \in T_x U$ is given by

$$V_x(f) := \frac{d}{dt}\Big|_{t=0} f(x + tV).$$

Exercise 4.9

Let $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$ be open sets and $\Phi \in C^\infty(U, V)$. Under the identification $TU \cong U \times \mathbb{R}^n$ and $TV \cong V \times \mathbb{R}^m$, show that the tangent map $T\Phi : TU \to TV$ is given by

$$T\Phi(x, v) = (\Phi(x), D\Phi(x)\, v) \quad \forall x \in U, v \in \mathbb{R}^n,$$

where $D\Phi(x)$ is the usual derivative, that is

$$D\Phi(x) \cdot v = \frac{d}{dt}\Big|_{t=0} \Phi(x + tv)$$

Local Coordinates on TM

Given an atlas $\mathcal{A} = \{(U_\alpha, \varphi_\alpha) \mid \alpha \in J\}$ on a manifold M, the tangent maps

$$T\varphi_\alpha : TU_\alpha \longrightarrow T\varphi(U_\alpha) \overset{\cong}{\uparrow} \varphi(U_\alpha) \times \mathbb{R}^n$$

identification from Exercise 4.8

are bijective (see Exercises 4.6 and 4.7), which allows us to represent a tangent vector locally as a pair

$$(x, v) = (x^i, v^i) \in \varphi(U_\alpha) \times \mathbb{R}^n$$

of n-dimensional vectors. In this way the pair $(TU_\alpha, T\varphi_\alpha)$ defines a chart on M.

Since we also have that

$$TM = \bigcup_{\alpha \in J} TU_\alpha$$

and

$$T\varphi_\beta \circ (T\varphi_\alpha)^{-1} = T\varphi_\beta \circ T\varphi_\alpha^{-1} = T(\varphi_\beta \circ \varphi_\alpha^{-1}) \in C^\infty\left(\varphi_\alpha(U_\alpha \cap U_\beta), \varphi_\beta(U_\beta \cap U_\alpha)\right)$$

by Exercise 4.7 · by Proposition 4.5 · by Exercise 4.9

for all $\alpha, \beta \in J$, it follows

$$T\mathcal{A} := \{(TU_\alpha, T\varphi_\alpha) \mid \alpha \in J\}$$

defines an atlas on M providing it with the structure of a $2n$-dimensional smooth manifold.

Definition 4.10 A vector bundle of rank k over a manifold M is a pair (E, π) consiting of a manifold E and a smooth map

$$\pi : E \longrightarrow M$$

such that for each $p \in M$, there exist an open neighborhood U of p and a diffeomorphism

$$\varphi : \pi^{-1}(U) \longrightarrow U \times \mathbb{R}^k,$$

which, when restricted to the fibre

$$E_p := \pi^{-1}(p),$$

defines a bijective map

$$\varphi|_{E_p} : E_p \longrightarrow \{p\} \times \mathbb{R}^k.$$

Note: The bijection $\varphi|_{E_p}$ endows E_p with the structure of a vector space.

Terminology The map $\varphi : E_U := \pi^{-1}(U) \longrightarrow U \times \mathbb{R}^k$ is called a (local) trivialization of E over U.

The pair (E_U, φ) is referred to as a vector bundle chart.

Theorem 4.11 (Inverse Function Theorem)

Suppose $f \in C^\infty(M,N)$ and $p \in M$. Then

$$T_p f : T_p M \longrightarrow T_{f(p)} N$$

is a (linear) isomorphism if and only if there exists a neighborhood U of p such that

$$f|_U : U \subset M \longrightarrow f(U) \subset N$$

is a diffeomorphism.

Proof Inverse Function Theorem from calculus.

Definition 4.12 Let $f \in C^\infty(M,N)$ be a smooth map. Then f is called

(i) a submersion at p if $T_p f : T_p M \to T_{f(p)} N$ is surjective, (This is also known as a regular point)

(ii) an immersion at p if $T_p f : T_p M \to T_{f(p)} N$ is injective, and

(iii) a local diffeomorphism at p if $T_p f : T_p M \to T_{f(p)} N$ is an isomorphism.

We further say that f is

(i') a submersion if it is a submersion at every $p \in M$,
(ii') an immersion if it is an immersion at every $p \in M$, and
(iii') a local diffeomorphism if it is a local diffeomorphism at every $p \in M$.

Finally, we say that $p \in M$ is a critical point for f if f is not a submersion at p (i.e. p is not a regular point of f.)

Example 4.13

(i) The map

$$f : \mathbb{R} \longrightarrow S^1 \subset \mathbb{C} : x \longmapsto e^{ix}$$

is a local diffeomorphism, but not a diffeomorphism.

(ii) The map

$$f : \mathbb{R}^2 \longrightarrow S^1 \subset \mathbb{C} : (x,y) \longmapsto e^{ix}$$

is a submersion.

(ii) The map

$$f : \mathbb{R} \longrightarrow S^1 \times S^1 \subset \mathbb{C}^2 : x \longmapsto (e^{ix}, 1)$$

is an immersion

Theorem 4.14

Suppose $f \in C^\infty(M,N)$, $p_0 \in M$, and f is a submersion at p_0. Then there exists a coordinate chart (V,ψ) containing $f(p_0)$, and a coordinate chart (U,φ) containing p_0 such that $\varphi(U) \subset V$ and the local representation $f_{\psi\varphi} = \psi \circ f \circ \varphi^{-1}$ is given by

$$f_{\psi\varphi}(x^1,\ldots,x^n,x^{n+1},\ldots,x^m) = (x^1,\ldots,x^n)$$

for all $(x^1,\ldots,x^m) \in \varphi(U)$.

Proof Implicit Function Theorem. ▨

Theorem 4.15

Suppose $f \in C^\infty(M,N)$, $p \in M$, and f is an immersion at p_0. Then there exists a coordinate chart (V,ψ) containing $f(p_0)$, and a coordinate chart containing p_0 such that $\varphi(U) \subset V$ and the local representation $f = \psi \circ f \circ \varphi^{-1}$ is given by

$$f_{\psi\varphi}(x^1,\ldots,x^m) = (x^1,x^2,\ldots,x^m,0,\ldots,0)$$

for all $(x^1,\ldots,x^m) \in \varphi(U)$.

Proof Implicit Function Theorem. ▨

Corollary 4.16 (Local normal form for submersions).

Suppose $f \in C^\infty(M,N)$ is a submersion and $p_0 \in M$. Given any coordinate chart (V,ψ) containing $f(p_0)$, there exists a coordinate chart (U,φ) containing p_0 such that the local representation $\hat{f} = \psi \circ f \circ \varphi^{-1}$ is given by

$$\hat{f}(x^1,\dots,x^n,x^{n+1},\dots,x^m) = (x^1,\dots,x^n)$$

for all $(x^1,\dots,x^m) \in \varphi(U)$.

Proof Follows directly from Theorem 4.14. ▨

Corollary 4.17 (Local normal form for immersions)

Suppose $f \in C^\infty(M,N)$ is an immersion and $p_0 \in M$. Given any coordinate chart (V,ψ) containing $f(p_0)$, there exists a coordinate chart (U,φ) containing p_0 such that the local representation $\hat{f} = \psi \circ f \circ \varphi^{-1}$ is given by

$$\hat{f}(x^1,\dots,x^m) = (x^1,x^2,\dots,x^m,0,\dots,0)$$

for all $(x^1,\dots,x^m) \in \varphi(U)$.

Proof Follows directly from Theorem 4.15 ▨

Theorem 4.18 Suppose $f \in C^\infty(M,N)$ is a submersion. Then each level set $f^{-1}(q)$, $q \in N$, is a submanifold of dimension $\dim M - \dim N$.

Proof (Hw: Provide a proof. Use Theorem 4.16)

Theorem 4.19 Suppose $f \in C^\infty(M,N)$ is an immersion. Then for each $p_0 \in M$, there exists a neighborhood U of p_0 such that $S = f(U)$ is a submanifold of dimension $\dim N - \dim M$.